actually of no use so long as the elements of weakness are present in the other parts subjected to compression.

To remedy these defects, it is proposed to rivet strong rings of angle iron at intervals along the flue—thus practically reducing its length, or in other words increasing its strength to a uniformity with that of the exterior shell. This alteration in the existing mode of construction is so simple, and yet so effective, that its adoption may be confidently recommended to the attention of all those interested in the construction of vessels so important to the success of our manufacturing system, and yet fraught with such potent elements of disaster when unscientifically constructed or improperly managed.

II. "On some Remarkable Relations which obtain among the Roots of the Four Squares into which a Number may be divided, as compared with the corresponding Roots of certain other Numbers." By the Rt. Hon. Sir FREDERICK POLLOCK, F.R.S., Lord Chief Baron. Received April 26, 1858.

(Abstract.)

The first property of numbers mentioned in this paper is best illustrated by an example—

$$13^2 = 169 \qquad 15^2 = 225.$$

These odd numbers may be divided into 4 squares, and the roots may be so arranged that they will have this relation to each other: the middle roots will be the same, and the exterior roots will be, the one 2 more, the other 2 less than the corresponding roots of the other. Putting the roots below the number and comparing them, the result is obvious.

169	225
0,3,4,12	−2,3,4,14
−2,4,7,10	−4,4,7,12
−4,5,8,8	−6,5,8,10
−6,4,9,6	−8,4,9,8

Each of the numbers may be divided into 4 squares in 4 different ways with this result, that the two middle roots of each are the same; and as to the exterior roots they differ by 2, the one being 2 more

the other 2 less than the corresponding roots of the other. So comparing 15^2 with 17^2,

225	**289**
4,3,10,10	6,3,10,12
6,5,10,8	8,5,10,10

the result is the same; and it is true of all adjoining odd squares. The paper contains a Table of odd squares (up to 27^2), compared in this manner with the odd square immediately before it and after it.

It is then shown that the same property continues when the 2 odd squares are increased by any the same even number—

49	**81**
0,2,3,6	−2,2,3,8
51	**83**
−1,3,4,5	−3,3,4,7

and also when they are (within certain limits) diminished by the same even number. It is then shown that a similar property belongs to the even squares +1, as seen below,

16 + 1 = 17	**36 + 1 = 37**
+1,0,0,4	−1,0,0.6
0,2,2,3	−2,2,2,5
37	**65**
−1,2,4,4	−3,2,4,6

and also to these numbers increased or decreased by the same even number.

If, instead of comparing the adjoining squares, the alternate squares be compared, a similar result is obtained; the middle roots are the same, the exterior roots differ by 4 instead of 2.

The proof of this property depends upon a general property of all odd numbers and upon a general theorem.

The property of odd numbers is this, that every odd number can be divided into 4 squares in such manner that 2 of the roots will be equal, 2 will differ by 1, 2 will differ by 2, &c. as far as the number is capable (from its magnitude) of having roots large enough to form the difference required: thus in the No. 39 there cannot be roots having a difference of 9; for the least number that can have that difference is $41=4^2+5^2$ and -4 and 5 differ by 9; but $39=1^2+2^2+3^2+5^2$, and the difference between -3 and 5 is 8; and the numbers 1, 2, 3, 5, either as positive or negative, give all the differences up to 8, but they do not give 2 equal roots: 39 is however divisible

into $1^2+1^2+1^2+6^2$, and then the equal roots are discovered. It is proved from the known properties of numbers that this property of having 2 roots whose difference will be 0, 1, 2, 3, &c., as far as is possible, belongs to all odd numbers. A new symbol is then suggested to represent the division of a number into 4 squares, such that 2 of the roots will have a given difference, and these are made the exterior roots; the number or figure denoting the difference is placed above on the left hand: thus $^{4}25$ denotes 0, 0, 3, 4 or −2, 1, 4, 2; $^{1}25$ denotes 1, 2, 4, 2.

The general theorem is this:—If any odd number of odd numbers be in arithmetical progression (4 being the common difference), as 9, 13, 17, 21, 25, then if the common difference be assumed as the index of the difference of roots to the middle term, and the higher terms in the series have as indices (4+1), (4+2), &c. in succession, and the lower terms have as indices (4−1), (4−2), &c., the series with its indices will be

2	3	4	5	6
9	**13**	**17**	**21**	**25**

and if the terms less than the middle term be divided into 4 squares with exterior roots having the differences indicated by their respective indices thus,

2	3	4
9	**13**	**17**
0,1,2,2	−1,2,2,2	−2,0,3,2

then the terms greater than the middle term will have this relation to the terms less than the middle term; the terms equidistant from the middle term will have their middle roots the same, and the differences of the exterior roots will increase; those nearest the middle term will have a difference of 1, the next 2, and so on, thus:

2	3	4	5	6
9	**13**	**17**	**21**	**25**
0,1,2,2	−1,2,2,2	−2,0,3,2	−2,2,2,3	−2,1,2,4

An algebraic proof is then given as to a series whose middle term is n and common difference p; and as n may be odd or even, and p also, and the index of differences may be *minus* as well as *plus*, the theorem applies frequently to even numbers, but not universally. The following example is given of the theorem applied to 17 terms of a series whose first term is 25, and common difference 1:—

−7	−6	−5	−4	−3	−2	−1	0	
25	**26**	**27**	**28**	**29**	**30**	**31**	**32**	
4,0,0,−3	5,0,0,−1	5,1,1,0	2,2,4−2	5,0,0,2	3,2,4,1	3,3,3,2	0,4,4,0	1
								33
9	8	7	6	5	4	3	2	0,4,4,1
41	**40**	**39**	**38**	**37**	**36**	**35**	**34**	
−4,0,0,5	−2,0,0,6	−1,1,1,6	−3,2,4,3	1,0,0,6	0,2,4,4	1,3,3,4	−1,4,4,1	

Comparing the terms above with the terms below, it is manifest the terms of the series are divisible into 4 squares whose roots conform to the law of the theorem. It is then shown that the odd squares, and also all the numbers mentioned in the beginning of the paper, can be made terms in an arithmetic series, and will therefore have the property stated. It is then suggested that the properties of numbers stated in the paper may have been in some form a portion of the mysterious properties of numbers by which Fermat announced he could prove his celebrated theorem of the polygonal numbers.

A Postscript was added, dated 20th May, which is here given entire.

Since this paper was sent to the Society, some other theorems of a similar kind have occurred to me, in which the terms of a series (*not* arithmetical of the 1st order) have a similar relation with regard to the roots of the 4 squares of which they may be composed, that is, those which are equidistant from the middle, or the middle term (according as the number of terms is even or odd), have the middle roots the same, and the exterior roots have an arithmetical relation to each other (varying with the distance from the centre), viz. the one being less and the other greater by the same quantity.

Thus, if any number of terms (exceeding 3) of either of the 2 series above-mentioned (viz. 1, 3, 9, 19, &c., or 1, 5, 13, 25, &c.), and, beginning with the first term, the differences be added "*inverso ordine*," a new series will be obtained possessing the property in
0 2 4 6 8
question; thus the first 7 terms of the 1st series are, 1, 3, 9, 19, 33,
10 12
51, 73; the differences are, 2, 6, 10, 14, 18, 22; if the differences be added "*inverso ordine*," the series becomes 1, 23, 41, 55, 65,

71, 73, each term of which may be divided into 4 squares, whose roots will be as follows :—

0	2	4	6	8	10	12
1	**23**	**41**	**55**	**65**	**71**	**73**
0,0,1,0	+1,2,3,3	0,3,4,4	−3,1,6,3	−2,3,4,6	−3,2,3,7	−6,0,1,6
		+2,0,1,6	−1,2,5,5	0,0,1,8		
			+1,1,2,7			

Here there is a middle term, all the terms equidistant from it have the same middle roots, the terms next to the middle term have the exterior roots, the one 2 less, the other 2 more, those next but one 4 less and 4 more, and the extreme terms 1 and 73 have their exterior roots one 6 less and the other 6 more than the corresponding roots of the other.

If 8 terms of the series be taken as 1, 3, 9, 19, 33, 51, 73, 99, and the differences be added "*inverso ordine,*" the series becomes
0 2 4 6
1, 27, 49, 67, &c., the terms of which divided into 4 squares, so that the differences of the exterior roots may correspond with the index, will be

0	2	4	6	8	10	12	14
1	**27**	**49**	**67**	**81**	**91**	**97**	**99**
0,0,1,0	−1,3,4,1	−2,4,5,2	−3,0,7,3	−4,0,7,4	−5,4,5,5	−6,3,4,6	−7,0,1,7
		+2,0,3,6	−1,4,5,5	−2,4,5,6	−1,0,3,9		
			+1,1,4,7	0,1,4,8			

Here there is no middle term ; the terms equidistant from the centre have the same middle roots, while the differences between the exterior roots increase as the numbers 1, 3, 5, 7.

The other series, 1, 5, 13, 25, &c., gives a similar result. If the differences of the first 7 terms be added "*inverso ordine,*" the new series, with its indices and the roots of the 4 squares which compose each term, will be as follows :—

1	3	5	7	9	11	13
1	**25**	**45**	**61**	**73**	**81**	**85**
0,0,0,1	0,0,4,3	−2,4,4,3	−3,0,6,4	−4,4,4,5	−4,0,4,7	−6,0,0,7
	−1,2,4,2	0,2,4,5	−2,4,4,5	−2,2,4,7	−5,2,4,6	
		+1,2,2,6		−1,2,2,8		

Here there is a middle term ; the equidistant terms have the same middle roots, the exterior roots are (next to the middle term) the

one 2 more, the other 2 less, and the differences increase by 2 as the terms are more distant from the middle term.

If the number of terms be 8, the resulting series with its indices and roots will be—

1	3	5	7	9	11	13	15
1	**29**	**53**	**73**	**89**	**101**	**109**	**113**
0,0,0,1	0,2,4,3	−2,2,6,3	−1,0,6,6	−2,0,6,7	−5,2,6,6	−5,2,4,8	−7,0,0,8
	+2,0,0,5	−1,0,6,4	+1,2,2,8	0,2,2,9	−4,0,6,7	−3,0,0,10	
		+1,0,4,6			−2,0,4,9		
		+2,0,0,7			−1,0,0,10		

and the differences of the exterior roots will be 1, 3, 5, 7. The reason of these results is, that the equidistant terms are always equal to the original corresponding term in the series increased by the same number.

Thus, in the first example, if to the terms

1, 3, 9, 19, 33, 51, 73 there be added
0, 20, 32, 36, 32, 20, 0, the result is
1, 23, 41, 55, 65, 71, 73, which is the

series with the differences added "*inverso ordine*." And in the last example, if to

1, 5, 13, 25, 41, 61, 85, 113 there be added
0, 24, 40, 48, 48, 40, 24, 0, the result is
1, 29, 53, 73, 89, 101, 109, 113, that is,

the series arising from the differences being added "*inverso ordine*."

It is worthy of observation that these numbers, 0, 24, 40, 48, 48, 40, 24, 0, which, added to the first 8 terms, produce a series identical with the result of the differences being added "*inverso ordine*," have the same effect upon any other consecutive 8 terms of the series. Take the 2nd term as the 1st of 8 terms—

5, 13, 25, 41, 61, 85, 113, 145, to these add
8 12 16 20 24 28 32

0 24 40 48 48 40 24 0, the result is

5, 37, 65, 89, 109, 125, 137, 145,
32 28 24 20 16 12 8

in which last series the differences are reversed or added "*inverso ordine.*" The appropriate roots of these numbers are—

3	5	7	9	11	13	15	17
5	**37**	**65**	**89**	**109**	**125**	**137**	**145**
−1,0,0,2	−1,2,4,4	−3,2,6,4	−2,0,6,7	−3,0,6,8	−6,2,6,7	−6,2,4,9	−8,0,0,9
	+1,0,0,6	−2,0,6,5	0,2,2,9	−1,2,2,10	−5,0,6,8	−4,0,0,11	
		0,0,4,7			−3,0,4,10		
		+1,0,0,8			−2,0,0,11		

which may be immediately obtained from the former series, the middle roots being the same; and the exterior roots, one of them one less, the other one more. In this way any consecutive 8 terms, with the differences reversed, may be each divided into 4 squares throughout the whole series.

And the same is true of 4 terms, 5 terms, or any number of terms. If 3 terms have the differences reversed, the numbers added are—

			0	4	0		
If 4 terms			0 8		8 0		
If 5 terms		0	12	16	12	0	
If 6 terms	0	16	24		24	16	0
If 7 terms	0 20	32	36		32	20	0
&c.			&c.		&c.		

The law under which these numbers are formed is obvious enough. The same numbers exactly are to be added to the other series (1, 3, 9, 19, &c.) to produce the same result.

If the 2 series be blended together, thus

0	1	2	3	4	5	6	7
1	1	3	5	9	13	19	25,

&c., the differences will be 2, 2, 4, 4, 6, 6, 8, 8, &c.; and if an odd number of terms be taken (so as to begin and end with a number from the same series), and the differences be added *inverso ordine,* a similar result occurs. Take 11 terms.

1 3 5 9 13 19 25 33 41 51 61, and add the differences "*inverso ordine,*" the series becomes, with its indices and roots,—

1	2	3	4	5		
1	**11**	**21**	**29**	**37**		
0,0,0,1	+1,0,1,3	−1,0,4,2	0,2,3,4	+1,0,0,6	6	
				−1,2,4,4	**43**	(middle term.)
11	10	9	8	7	−3,3,4,3	
61	**59**	**57**	**53**	**49**		
−5,0,0,6	−3,0,1,7	−4,0,4,5	−2,2,3,6	0,0,0,7		
				−2,2,4,5		

In this case the additions are 0, 8, 16, 20, 24, 24, 20, 16, 8, 0; and if these be added to any other consecutive 11 terms (the 1st term having an *odd* index), they produce the same effect as if the differences were reversed; and the resulting numbers have the property of the terms equidistant from the centre, being connected by their roots, having the relation so frequently mentioned. It may be further remarked, that the numbers produced by reversing the differences are the initial numbers from which, by adding 2, 2, 4, 4, &c., 61 may be formed of the squares, which make the differences of its exterior roots 10, 9, 8, &c.

1	2	3	4	5	6	7	8	9	10
11	**13**	**15**	**19**	**23**	**29**	**35**	**43**	**51**	**61**
	2	2	4	4	6	6	8	8	10
0,0,3,1	0,0,3,2		−1,0,3,3		−2,0,3,4		−3,0,3,5		−4,0,36

1	2	3	4	5	6	7	8	9
21	**23**	**25**	**29**	**33**	**39**	**45**	**53**	**61**
	2	2	4	4	6	6	8	8
0,2,4,1								−4,2,4,5

and so of all the others.

The matter referred to in this Postscript tends to strengthen the suggestion already made, that the properties of numbers referred to are connected with the mysterious and abstruse properties to which Fermat referred as enabling him to prove the theorem he announced of the Polygonal Numbers.

III. "Observations on the Mer de Glace."—Part I. By John Tyndall, Ph.D., F.R.S. &c. Received June 10, 1858.

(Abstract.)

In this paper the author communicates the first part of a series of observations upon the Mer de Glace, made during a residence of six weeks at the Montanvert last summer *. He corroborates the laws regarding the swifter flow of the central portions of the ice-stream, first established by Prof. Forbes, and shows how the velocity changes as the width of the glacier varies. The Mer de Glace moves through a valley which twice turns a convex curvature to the east, and once to the west. The points of swiftest motion at these curves are found to be not central, but thrown to that side

* During the whole of which period he was most ably assisted by his friend Mr. T. A. Hirst.